

Introduction to Abstract Algebra

Practice Midterm 1, March 17, 2026

1. Show that the binary operation

$$a * b := \frac{a + b}{ab}$$

is not a group operation on $\mathbb{R}_{>0}$.

2. Show that the binary operation

$$a * b := \frac{a + b}{1 - ab}$$

defines a group on the open interval $G := (-1, 1)$. Compute explicitly the identity element and the inverse a^{-1} of any $a \in G$.

3. Use the function

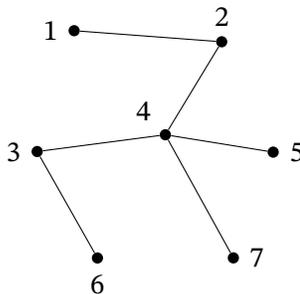
$$f(x) = \frac{1 + x}{1 - x}$$

to show that the group $G := (-1, 1)$ with the binary operation of the previous exercise is isomorphic to the group $(\mathbb{R}_{>0}, \cdot)$.

4. Consider the subset

$$T := \{(12), (24), (34), (36), (45), (47)\} \subseteq S_7,$$

so that each element of T is a transposition. We can use the set T to draw a graph by connecting a and b with an edge if $(ab) \in T$. This way, we obtain:



- a) In the above graph, there is a path from 2 to 6. Use this path to show that the transposition (26) is an element of the subgroup $\langle T \rangle$ generated by T .
- b) Deduce that every transposition (ab) lies in the subgroup generated by T , and conclude that $\langle T \rangle = S_7$.

Hints

1. Suppose that e is an identity element. Derive a contradiction.
2. Show first that the operation is well-defined, meaning that we have

$$-1 < \frac{a+b}{1-ab} < 1$$

for every $a, b \in (-1, 1)$. Then prove the group axioms. For a clear proof, state which axiom you are about to prove before proving it.

3. Show that this function is a bijection $(-1, 1) \rightarrow (0, \infty)$ and preserves the group operation.
4. In part (a), use that $(ab)(bc)(ab) = (ac)$. In part (b) observe that there is a path between any two points, so part (a) can be generalized to conclude that every transposition is in $\langle T \rangle$. Since the transpositions generate the symmetric group, it follows that T generates the symmetric group.

General observation: If a set T of transpositions determines a connected graph on the set $\{1, \dots, n\}$, then T generates S_n .